

Mathematical Platonism: from objects to patterns

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The aim of this paper is to reveal the tacit assumptions of the logicist and structuralist theories on the nature of cardinal numbers. The privileged background theory is the representational theory of language with its main conjecture about necessary correspondence between metaphysical structure of reality and logical structure of language. In such a theory of language we meet a number of thesis on relations between syntax and semantics. Special attention is paid to the so called ‘syntactic priority thesis’: if an expression has the role of a singular term in a true sentence then there must be an object denoted by that term. On these assumptions logicist develops the view that numerals refer to numbers and that numbers must be a kind of abstract objects. That position we call ‘Platonism of objects’. The inherent weakness of ‘Platonism of objects’ gave rise to the structuralist theory of cardinal numbers. Although the structuralist position in philosophy of mathematics is more congenial to the nature of mathematics as a deductive science, its proponents were reluctant to abandon assumptions on the relations between syntax and semantics as conceived in the representational theory of language. After examining some philosophical implications of measurement theory and intensional logic, we argue for a non-representational theory of language. In such a theory, logical and mathematical structures are not conceived as common form of reality and language, but rather as structures describing a multitude of ‘semantical spaces’ inherent in the language.

Truth and the principle of compositionality

Our understanding of the way language functions employs a number of often un-reflected presumptions. Some of them are more fundamental than the others. An example of such a presumed principle is the one that states that a well-formed sentence has a meaning. The role of such a principle is methodological: encountering a seemingly well-formed sentence, which fails to have a meaning, leads to reexamination of the formation rules. The principles that deserve special attention include the principle which correlates syntactic role of the expressions within a sentence with their interpretation, and the principle which states that the meaning of the compound expression depends on the meanings of its parts. The first principle is sometimes referred to as ‘syntactic priority thesis’, the second as ‘principle of compositionality’. By adopting these principles we also seem to be forced to adopt a class of abstract entities. Although there are no philosophical reasons preventing us from accepting abstract entities in the ontological inventory, still there are strong epistemological reasons against accepting a purported object that can not be individuated. The philosophical investigation into the nature of cardinal numbers was conducted in the framework of ‘standard semantic approach’,

which employs correspondence theory of truth, principle of compositionality and syntactic priority thesis. The multitude of *ad hoc* invented solutions for the problem of meaning of arithmetic sentences shows the limits of standard semantic approach.

The implicit discovery of the principle of compositionality may be attributed to Plato, who in *Theaetetus* posed the problem of falsehood. If the truth of a sentence consists in its correspondence to a fact then false sentence is about nothing. The solution proposed in *Sophist* relies on the distinction between subject and predicate: a sentence can be about something (subject) and yet be false if the predicate does not apply to the subject. It seems that view according to which the meaning of a sentence consists in its truth-value leads to the acceptance of the principle of compositionality. More recent example can illustrate the point:

But facts or states of affairs have never been shown to play a useful role in semantics...This is not surprising, since there is a persuasive argument, usually traced to Frege (in one form) or Kurt Goedel (in another), to the effect that there can be at most one fact or state of affairs."

The dependence of the meaning of the whole on the meanings of the parts (and their mode of combination) can be expressed within formal semantics. Our example will be drawn from the language of quantificational logic without free variables. Elementary sentence in that language has the form 'an object has a property' or 'objects stand in a relation'. There is semantical difference between an object from the domain and subset of the domain corresponding to syntactical difference between predicate (general term) and subject (singular term), respectively. The model (D, f) (in the wider sense of the term) consists of a set D of objects and interpretation function f . In the propositional logic interpretation is defined as valuation v which assigns a truth value for each well-formed sentence. We can show the way in which the truth (*i.e.* meaning) depends on the meaning of its parts with the aid of symbols of formal semantics. The sample for analysis has been taken from Davidson:

"‘Russell is a philosopher’ is true if and only if the object named by ‘Russell’ belongs to the class determined by the predicate ‘is a philosopher’"

In formal notation valuation v_M of a proposition in a model $M=(D, f)$ may be connected to interpretation f in the following way (with a standing for a name and P standing for a predicate): $v_M(P(a))=T$ iff $f(a) \in f(P)$. The disquotation scheme (T-convention 'v(P(a))=T iff P(a)') relativized to a model allows substitution with 'P(a)' for ' $v_M(P(a))=T$ ', and we arrive at 'P(a) iff $f(a) \in f(P)$ '. The standard semantic approach accepts correspondence theory of truth in a relativized way as 'truth in a model' and explicates truth valuation with the aid of principle of compositionality. The syntactic priority thesis then follows immediately. Interpretation function f assigns exactly one object from the domain D to each singular term t , or $f(t) \in D$. Therefore, if an expression belongs to syntactic category of singular terms, its interpretation is an object. The standard semantic approach is completely extensional. It takes truth value as original semantic value and explains it with the extensional semantical values of its parts and their mode of combination. Carnap made twofold division of semantical values in extension and intension, The syntactic categories are predicates, singular terms and sentences and their extensions (extensional semantical values) are classes, objects and truth-values. 'Syntactic priority thesis' makes

explicit the tacit assumptions of representational theory of language concerning relations between syntactic categories and extensional semantic values. The thesis as applies to singular terms can be stated ‘If an expression belongs to syntactic category of singular terms and it occurs in a true sentence then there exists an object to which it refers’. The ‘ontological commitment’ involved in fixing a range for variable assignment is still another way of telling the same.

In the standard semantical approach the set theory provides the formal model for the language. It comes as no surprise that among axioms and axiom schemata of the set theory we can find those that guarantee the existence of a class consisting of all objects that satisfy a condition. So-called ‘axiom of comprehension’ or ‘axiom schema for class formation’ correspond to generalized ‘syntactic priority thesis’ applied to predicates. We may therefore state that if a predicate applies to singular terms then there is a class of objects denoted by that predicate. The latter application of syntactic priority thesis seems less intuitive since sets are arbitrary objects. In the postscript for the second volume of *Grundgesetze der Arithmetik* Frege expressed the doubt concerning self-evidence of that principle:

“It is a matter of my Axiom (V). I have never disguised from myself its lack of self-evidence that belongs to other axioms and that must properly be demanded of a logical law.”

The extensional approach to the issue of ‘meaning of meaning’ rises an interesting question: does set theory with axiom of extension (informally, a set is completely determined by its membership) explicate the assumptions of representational theory of language or representational theory by accepting set-theoretic semantic model adopts at the same time the axioms of set theory. Anyhow, it is clear that inter-theoretical relations between logic and ontology became mediated through mathematics.

Once accepted, the representational theory of language with standard semantic approach lead to the ontologization of purported referents of singular terms. In philosophical considerations on nature of cardinal numbers the acceptance of numerals as singular terms lead to ontological thesis that numbers are objects. The only way out from this hypostatization would be found if logical role of numerals could be connected to some other syntactic category.

What numerals are

In the language of first-order quantificational logic the following types of expressions can be distinguished: individual constants, predicate constants, propositional constants, function constants, variables, connectives and quantifiers. Individual constants and functional expressions belong to syntactic category of singular terms. The answer which first comes to mind is that logical role of numerals is similar to the role of quantifiers. The sentence stating that n objects from the domain with m members satisfy predicate P is equivalent to a disjunction. In that disjunction (with exactly $m!/(m-n)!n!$ disjuncts) where each disjunct is one among combinations of conjunctions with exactly n affirmative and $(m-n)$ negative sentences. Numbered quantifiers can be easily transformed into not numbered as follows: $\exists_0 xPx$ for $\neg \exists xPx$; $\exists_1 xPx$ for $\exists x \forall y (Py \leftrightarrow$

$x=y$); $\exists_2 xPx$ for $\exists x \exists y (x \neq y \wedge \forall z (Pz \leftrightarrow (z=x \vee z=y)))$, $\exists_{n+1} xPx$ for $\exists x (Px \wedge \exists_n y (Py \wedge x \neq y))$. Still there are important obstacles for treating numerals as quantifiers. To mention a few: there's no translation for infinite domain, there is no translation for sentences about quantitative relations (for instance, there are 'as many', 'more than', 'less than')

We could treat numerals as predicates in second order logic as Frege did.

"If I say "the King's carriage is drawn by four horses," then I assign the number four to the concept "horse that draws the King's carriage"

The notation would be (where H stands for "horse that draws the King's carriage"): $4(H)$ and this second-order statement can be translated into $\exists_4 xHx$. The problem concerning quantitative relations persists although some of them can be defined by unqualified relation of equinumerosity (say R). For instance, 'There are as many Fs as Gs' is expressible as ' $\exists R R(F,G)$ ', while 'There are twice as many Fs as Gs' is not expressible without recurrence to numbers. It follows that we must treat numerals as singular terms in order to make possible reference to numbers. If numerals are singular terms then they are either individual constants (or 'proper names' in Russellian sense) or functional expressions. Functions satisfy existence ($\forall x \exists y y=f(x)$) and uniqueness ($\forall x \forall y \forall z ((f(x)=y \wedge f(x)=z) \rightarrow y=z)$) conditions as a matter of definition. The possibility of denoting each and every object from a domain by a functional expression implies the existence of a total order defined on that domain. From epistemological point of view, 'successor of (successor of zero)', where 'successor' is a function and not a predicate, and 'two' are not the same concepts. Denoting by functional expressions excludes the possibility that a number can be known in isolation, while denoting by a 'proper name' leaves that possibility open.

If numerals are functional expressions then they are singular terms, but their complexity eliminates them as candidates for names in strong Russellian sense.

"A name is a simple symbol, directly designating an individual which is its meaning, and having this meaning in its own right, independently of all other words."

The other option is to regard numerals as names in Russellian sense. In that case there must be an individual for each numeral. Consequently, we encounter the problem of the nature of numbers as designations of numerals. We will disregard the issues concerning what kinds of objects there are. The kinds of entities that can apply for role of numbers are so-called "ideal" or "Platonic" entities.

"Candidates for entities independent of all real entities include such things as mathematical entities, concepts or properties on a Platonist conception, as they are said to be capable of existing in the absence of real entities."

Having a property seems to be the least condition for being an object. By Berkeley argument, the object with no properties is not object at all. If natural numbers are objects then they have some properties: not only all hereditary properties but also at least one distinguishing property belonging to each number as such. The distinction between logicism and structuralism position in the theory on the nature of cardinal numbers reduces to the question of existence of an intrinsic property, or to the closely related

question of the possibility of knowing an isolated number. If the numbers are objects in their own right then each of them must have an intrinsic property. Structuralism does not admit such properties for numbers, therefore no structuralist may accept numerals as names without abandoning standard semantic approach. On the epistemological side, competent and incompetent use of numerals by empirical subjects cannot give a conclusive answer regarding the question of syntactic category for numerals since there is more than one logical role a numeral can assume. It seems that there is common agreement that the project of ‘fixing a reference’ for numerals has failed. Theory of types is a way to save the axiom of comprehension, but not a way of ‘fixing of reference’, and it requires accepting of an ascending hierarchy of numbers denoted by the same numeral (beginning with the one from sets of sets of individuals upwards). Consequently, untyped numerals fail to be names.

Individuation and characterization

The main epistemological problem of *Platonism of objects* lies in the impossibility of independent individuation of number-objects. Empirical individuation is not possible *per definitionem*: since abstract entities have no spatio-temporal location and are causally inert. Intratheoretical individuation is not possible because axiomatic characterization of natural numbers can not give any intrinsic property. The defense of logicism stating that logic (including theory of sets) provides the individuation of numbers by their intrinsic properties is epistemologically implausible. By well-known Piagetian argument supposed generalization leading to formation of the concepts of a kind ‘set of equinumerous sets’ requires discarding of all the properties of members of latter sets. Since the object deprived of all properties is no longer an object, it follows that formation of (logicist) number concept requires a complementary operation (namely, serialization) in which objecthood is preserved thanks to endowing (property deprived) objects with extrinsic properties of having a position in a sequence. Resnik notes:

“The second (problem) arises from the fact that no mathematical theory can do more than determine its objects up to isomorphism. Thus the Platonist seems to be in paradoxical position of claiming that a given mathematical theory is about certain things and yet be unable to make any definitive statement of what these things are.”

The impossibility of intratheoretical individuation of numbers by their intrinsic properties can be illustrated by reflecting on Goedel incompleteness theorem. We will draw our attention to the question of how accurately can natural numbers be described. Goedel has proved that consistency implies incompleteness for the theories formulated within language rich enough to express its own syntax (and the language of elementary number theory is of that kind). To prove this fact he uses the arithmetization of syntax (to each sentence he assigns a unique number reflecting its syntactical properties) and the notion of (primitive) recursive function as formal way of defining provability. On that basis it is possible to construct a well-formed formula that informally may be understood as a sentence that speaks of itself, stating that it is not provable. Such a sentence g is true if it is not provable *et vice versa*. Since that sentence remains unprovable for any consistent set of axioms A it follows that two sets $\{A, g\}$ and $\{A, \neg g\}$ are both consistent and that, in turn, means that they both have structures satisfying them. If we use term model in the narrower sense we may say that these sets of sentences have different models. We say

that these models are not isomorphic. It follows that any axiomatization of elementary number theory will have at least two different models. Therefore, mathematics can describe its structures only “up to isomorphism.” Although Goedel declared himself as Platonist, we can use his incompleteness theorem as strong argument against Platonism of objects. There is no foundational need for postulating the existence of an intrinsic property for, say, numeral 2 and to take it to be a name for ‘ $\{ \alpha \mid \exists y \exists z \forall x (x \in \alpha \leftrightarrow ((x=y \vee x=z) \wedge y \neq z)) \}$ ’, or more generally to be a name 2_k for set_k of sets_{k-1} with exactly two members $_{k-2}$ (indexes indicating types). The other ‘objects’ (like $\{ \emptyset \{ \emptyset \} \}$ or $\{ \{ \emptyset \} \}$) may do equally well provided they are arranged in suitable progressions. Intratheoretical determination of numbers is therefore excluded.

A different, nonobjectual conception of numbers is being developed within structuralist philosophy of mathematics. Numbers are not objects endowed with intrinsic properties. They can not be given in isolation since they are deprived of all properties besides those belonging to them as elements of a structure. Benacerraf in his seminal paper defines structuralist thesis on nature of numbers:

“(…) numbers are not objects at all, because in giving the properties (that is, necessary and sufficient) of numbers you merely characterize an *abstract structure* – and the distinction lies in the fact that the ‘elements’ of the structure have no properties other than those relating them to other ‘elements’ of the same structure.”

In his influential paper Resnik follows him:

“The objects of mathematics, that is, the entities which our mathematical constants and quantifiers denote, are structureless points or positions in structures. As positions in structures, they have no identity or features outside of a structure.”

If the structuralist position is adopted then there are strong reasons to argue by contraposition on ‘syntactic priority thesis’. If numerals are singular terms then they refer to objects, namely numbers, but if numbers are not objects then neither numerals are singular terms. And since numbers are not objects, numerals are not singular terms. The adherents of structuralism were obviously reluctant to accept the conclusion. The pragmatic reason is obvious; the rules of inference apply to usual depiction of logical form, changing the latter implies revision of the former. Using Davidson’s metaphor, our interest in ‘logical geography’ determines ‘logical anatomy’, that it is to say, we choose formalization that justifies accepted inferential relations. Some proponents of structuralist position in philosophy of mathematics did not abandon singular termhood of numerals (Benacerraf), while others loosened the reference relation and took numerals to be general terms (White, Field, Resnik). There are infinitely many models for Dedekind-Peano axioms; any recursive progression is adequate. One possible reaction in philosophical semantics is to dismiss the question of reference of numerals as ill-defined. The purported ‘objects’ of the elementary number theory “do not do the job of numbers singly; the whole system performs the job or nothing does”. Therefore, the referent of a numeral does not do the job in virtue of its having an intrinsic property, but only in virtue of being ‘located’ in a structure and having only the properties belonging to it as an element of that structure. Any object can play the role of any number, provided it is adequately located in a progression. Within standard semantics the issue of reference may be rejected only at the cost of inconsistency involved in accepting numerals as singular terms and not accepting numbers as objects. Treating numerals as singular terms requires

abandonment of standard semantic approach. Alternative structuralist reaction is to allow for different kind of referring. The numerals ‘partially refer’ to appropriate elements in each progression. They are similar to singular terms as far as they refer to only one element in each progression, and similar to predicates as far as they apply to multitude of elements in different progressions. In White’s proposal ‘2’ is a binary predicate with a place for a progression p and a place for an element x in it. Numeral ‘2’ is to be read ‘ x is a 2 in p ’. The conditions for applicability of numeral-predicates are defined by the usual axioms of arithmetic and their applicability requires existence and uniqueness of elements. Therefore, it seems that treatment of numerals as functional expression would be more akin to the spirit of structuralism; with first element denoted by, say, ‘0(p)’ or ‘the first element in a progression p ’ or ‘zero function’ and all others defined by successor function. Read this way White-Field-Resnik proposal turns out to be a kind of Pythagoreanism since any object is a number if arranged in a suitable progression. The idea that arithmetical statements are properly understood as quantifying over functions is plausible since it does not require a privileged progression and Platonism of objects. On the other hand, the idea that arithmetical statements quantify over progressions and their elements seems vacuous since being a model of arithmetic does not depend on the nature of objects but only on their relations. It is impossible to determine an arithmetical relation in an extensional way by listing n -tuples of objects holding it. So it seems that semantics of mathematical sentences requires a special kind of analysis.

The same argument applies to neo-Fregean relativization of singular termhood to the context of discourse. It is true that the same expression can function in one context as singular, while in other as general term. For example: ‘Chess queen may move in all directions’ and ‘This figure is a chess queen’. We can talk about roles in the game and about objects taking the role using lexically the same expression, and yet giving it different syntactic role. The difference remains unreducible, for the meaning of general terms defined by their role can not be reduced to the set of objects playing that role. In considering semantics of arithmetic sentences we encounter an important fact concerning the types of description that can be given. From the onto-logical point of view accepting an object as *descriptum* implies the possibility of giving an individuating description. On the other hand, a structure is entirely different *descriptum*. We **characterize** structures, which may have multiple realizations or instantiations. The structures or patterns are being instantiated by objects and their relations, so we may say that objects may be individuated and that their relations may be characterized as an occurrence of a pattern.

Non standard semantics and modal truth of mathematics

Fruitful analogy for semantics of arithmetical sentences can be found in modal logic. Axioms of different modal logics characterize frames. For instance axiom $\Box p \rightarrow p$ characterizes class of reflexive frames. We could state the axiom otherwise as $\forall x \forall y (R(x, y) \rightarrow (Px \wedge Py))$ where R should be read as ‘accessibility relation’, and P as ‘verifies proposition p .’ The frames give semantics for intensional language. Figuratively speaking, they give valuation space for valuation of modal sentences, whose truth-value can not be determined by looking at one world only. Which worlds (i.e. place where non-modal part of modal sentence is being interpreted) are relevant depends on accessibility

relation. It does not make sense to pose question of interpretation of frames in the framework of standard semantics and assert on the basis of syntactic priority thesis that the one, who uses Kripke-semantics, is committing herself to the acceptance of an infinity of possible worlds. The position of linguistic pluralism seems to be philosophically sound. The axioms of modal logics characterize structures that give semantics for some sub-languages in the ‘language family’. Take an example: in deontic logic there is no place for axiom ‘ $\Box p \rightarrow p$ ’ of modal logic T. if we interpret ‘ \Box ’ as ‘it is obligatory that’. On the other hand, in logic of the language using physical vocabulary the axiom seems to be a truism if ‘ \Box ’ is interpreted as ‘it is implied by deterministic laws of nature’. Our thesis comes close to a number of non-standard semantics, like Gardenfors theory of conceptual spaces, dynamic semantics of Dutch logic school or theories of cognitive semantics, to mention a few. The main idea of our proposal can be summarized as follows: there is no universal correspondence between syntax, semantics and ontology such as one implied by representational theory of language. The semantics for expressions belonging to a syntactic category of singular terms depends on the mode of description. Having a structure for *descriptum* does not involve the existence of referents for singular terms used in characterizing it. The sentences used in characterization of a structure do not fit the schema of standard semantic analysis. Truth valuation in intensional languages is made possible by multiple valuations in a structure. Therefore we must distinguish sentences speaking about a thing, from sentences characterizing a structure that makes such speaking possible. The meaning of the latter kind may not be equaled to truth conditions since they have the job of providing a way of having a truth-value (conceived more broadly than in correspondence-to-facts theory) or semantic value, in general, for other sentences. Characterizing a structure does not impose ontological commitments. We are describing a possible structure, i.e. a one that may be instantiated, and elementary semantical valuation of being true in the world does not apply here.

On the other hand, logical form may not always be taken at the face value. We would formalize a sentence like “The mass of this book is 0.5 kilograms” as “mass-in-kg (this book, 0.5)”. Still, nobody thinks that there is a relation between the book and real number 0.5. This sentence is relational for sure, but the asserted relation is not the relation between a physical object and real number, on the contrary it is relation between an object and a structure. The theory of measurement shows that in measuring we use numerical structures in order to ascribe some relations defined on it onto measured empirical structure. To stick to the example, by using real numbers as measuring structure for mass measurement we are imposing the law of transitivity on that property. Semantics for measurement sentences can not be treated in the standard way. In measurement we are not arriving at usual relational statements, but at hidden nomic or general statements. On the contrary: measuring means transferring nomic relations defined on measurement space to empirical objects. The determination of ‘logical anatomy’ is not guided by ‘identity of logical form of language and reality’ but rather exhibits the accepted theory on the nature of objects and their properties. For convenience of writing we use “weight-in-kg (this book, 0.5)” whereas, if we stick to standard general semantics, we should use one amongst infinitely many monadic predicates of the form “weight-of $-n$ kg (x)” in which case ‘logical geography’ would become inexplicable.

Semantics of mathematical sentences differs from semantics of empirical sentences since mathematical structures may be used in empirical sentences as a way to represent empirical phenomena. Mathematical and logical structures are constitutive for semantics used in describing the way the world is. Truth in mathematics and logic is different from truth in empirical sciences. In empirical sciences being true means being true in the privileged model, model of reality. The source of privileging is a philosophical issue: theories concerning its source abound and are of no consequence here. Although being true and being provable do not necessarily coincide for the theories formulated within languages rich enough to express its own syntax, still Hilbert program survives in so far the axioms are that what really matters. All the models for Dedekind-Peano axioms are not isomorphic but being a theorem of arithmetic means to be true in **all** models of axioms. Therefore we may propose a modal definition for mathematical truth: $v(p)=T$ iff *p is true in each model in which axioms are true*. We end up with the following hierarchy: p is satisfiable iff there is a model in which p is true, p is real-theoretically true iff p is true in ‘the model of reality’, p is axiomatically true iff it is formulated in language of theory T and p is true in all models in which axioms of T are true, p is tautologically true iff p is true in all models.

Benacerraff posed the question:

“But what of sentences that are not composed (or correctly analyzable as being composed of) names, predicates and quantifiers? More directly to the point, what of sentences that do not belong to the kind of language for which Tarski has shown us how to define truth?”

The answer we are proposing does not violate Tarski requirement for relative truth. The truth of mathematical sentences is to be defined in the modal way. The semantics of expressions occurring in mathematical sentences cannot be equaled with semantics of empirical sentences. The language of mathematics allows for characterization of the structures, but it does not allow for individuation of collections of objects instantiating these structures. Standard semantic approach along with ‘ontological commitment’ is restricted to class of empirical sentences. Platonism as ontological thesis allowing for the existence of abstract entities does not seem to be philosophically acceptable. On the other hand, Platonistic epistemological thesis that requires existence of autonomous structures for the possibility of cognition seems to be completely justified. The mathematical truth is indeed necessary, but its necessity is necessity of a semantical space that makes empirical cognition possible.

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