Propositional Modal Logic

a Selection of its Properties, Applications, and Problems

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Overview

1. A historical introduction
2. Properties of modal language
3. Some applications and problems
   - Natural deduction in modal logic
   - Dynamic semantics and problem of consequence relation

One world and one relation between world and language

- The philosophy of first-order logic has been exposed in Wittgenstein's *Tractatus logico-philosophicus*.

Tractarian theory

The world		Natural science
⇓		⇑
Fact		True proposition
⇓		⇑
State of affairs		Proposition
⇓		⇑
Relation of objects		Predicate and names
▲	PICTURE RELATION	▲

⇓ shows deconstruction path.
⇑ shows construction path.

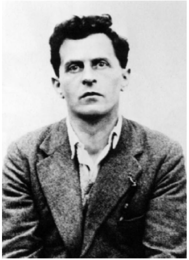
One world is not enough

- According to the *Tractarian* criterion, modal propositions are not propositions at all since they are not truth-functions.
- The “one world” semantic theory cannot accommodate modal propositions since the truth value of their “elementary propositions” in *The World* does not determine the truth-value of the modal compound.

Example

The truth of what is obligatory to be (ought to be) the case is logically independent of that what is the case.
Let **O** stand for ‘It is obligatory that’.
Both $p \wedge \mathbf{O}p$ and $\neg p \wedge \mathbf{O}p$ are satisfiable.

Tractatus logico-philosophicus



5 Propositions are truth-functions of elementary propositions.

. . .

6.42 Hence also there can be no ethical propositions.

. . .

7 Whereof one cannot speak, thereof one must be silent.

Leibniz and modal analysis of normative concepts

Modal approach to normativity

Licitum enim est, quod viro bono possibile est.
Debitum sit, quod viro bono necessarium est.[a]

Gottfried Wilhelm Leibniz.
Letter to Antoineu Arnauldu, November 1671.
Saemtliche Schriften Und Briefe. Zweite Reihe: Philosophischer Briefwechsel. Erster Band 1663–1685,
Berlin: Akademie Verlag.

[a] That is permitted what a good man possibly is.
That is obligatory what a good man necessary is.

Gottfried Wilhelm Leibniz
(1646.–1716.),
statue at University u Leipzigu

Analysis

In Leibniz's definition normative concepts (permission **P**, obligation **O**) are defined in terms of (i) alethic modalities (possibility $\Diamond$, neccessity $\Box$) (ii) normative properties (being a good man Gi).

$$\mathbf{O}\varphi \leftrightarrow \Box(\mathsf{Gi} \rightarrow \varphi)$$

$$\mathbf{P}\varphi \leftrightarrow \Diamond(\mathsf{Gi} \wedge \varphi)$$

Deontic logic as modal logic

Philosopher's recollection

One day when I was walking along the banks of the River Cam —I was at that time living in Cambridge (England)— I was struck by the thought that the modal attributes "possible," "impossible" and "necessary" are mutually related to one another in the same way as the quantifiers "some," "no" and "all." I soon found that the formal analogy between quantifiers and modal concepts extended beyond the patterns of interdefinability. . . I had made another accidental observation —this time in the course of a discussion with friends— namely that the normative notions of permission, prohibition, and obligation seemed to conform to the same pattern of mutual relatedness as quantifiers and basic modalities.

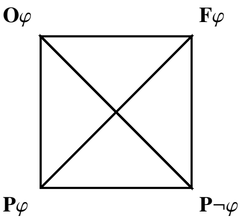
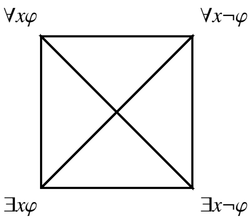
Georg Henrik von Wright.
Deontic logic: a personal view.
Ratio Juris, 12:26–38, 1999.

Ludwig Wittgenstein and Georg Henrik von Wright
(Photograph from April 1950.; taken in Von Wright's garden while Wittgenstein was a guest at his house.)

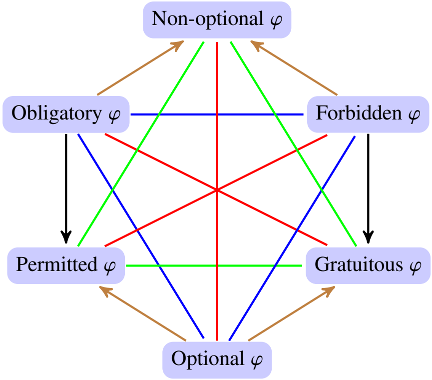
ANALOGY OF QUANTIFICATION AND MODALITY

Duality; square of oppositions

Quantifiers	Alethic modalities	Deontic modalities
$\forall x\varphi \quad (\neg\exists x\neg\varphi)$ ALL ...	$\Box\varphi \quad (\neg\Diamond\neg\varphi)$ NECESSARY ...	$\mathbf{O}\varphi \quad (\neg\mathbf{P}\neg\varphi)$ OBLIGATORY ...
$\exists x\varphi \quad (\neg\forall x\neg\varphi)$ SOME ...	$\Diamond\varphi \quad (\neg\Box\neg\varphi)$ POSSIBLE ...	$\mathbf{P}\varphi \quad (\neg\mathbf{O}\neg\varphi)$ PERMITTED ...
$\forall x\neg\varphi \quad (\neg\exists x\varphi)$ NO ...	$\Box\neg\varphi \quad (\neg\Diamond\varphi)$ IMPOSSIBLE ...	$\mathbf{F}\varphi \quad (\mathbf{O}\neg\varphi$, i.e. $\neg\mathbf{P}\varphi)$ FORBIDDEN ...

Modal logic	First-order logic
	

Hexagon of "oppositions"[1]

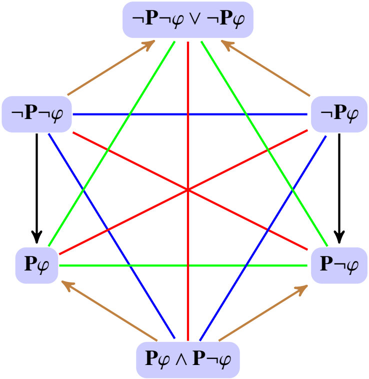


Four logical relations resulting from mutual definability (duality) of normative notions. The last one is D axiom

Name	Property	Symmet
Contrariety	Both sentences cannot be true.	Yes.
Subcontrariety	Both sentences cannot be false.	Yes.
Contradiction	Both sentences cannot be true, and both sentences cannot be false.	Yes.
Implication	It cannot be so that the source sentence is true and target sentence is false.	No.
Implication	It cannot be so that the source sentence is true and target sentence is false.	No.

[1] Some synonyms:[Permitted; Allowed][Optional; Allowed and non-obligatory][Gratuitous; Non-obligatory; Omissible][Forbidden; Prohibited; Impermissible][Non-optional; Obligatory or forbidden][Obligatory]

D AXIOM



Obligatory $\mathbf{O}\varphi \leftrightarrow \neg\mathbf{P}\neg\varphi \leftrightarrow \mathbf{F}\neg\varphi$

Forbidden $\mathbf{F}\varphi \leftrightarrow \neg\mathbf{P}\varphi \leftrightarrow \mathbf{O}\neg\varphi$

The "black arrow" implications $\mathbf{O}\varphi \rightarrow \mathbf{P}\varphi$ and $\mathbf{F}\varphi \rightarrow \mathbf{P}\neg\varphi$ are equivalent. [a] This implication is called D axiom. It can also be read as $\neg(\mathbf{O}\varphi \wedge \mathbf{F}\varphi)$,i.e., as a claim on contrariety of $\mathbf{O}\varphi$ and $\mathbf{F}\varphi$.
The analogy with alethic modalities holds since $\Box\varphi \rightarrow \Diamond\varphi$.

[a] Assuming modal congruence.

Axioms and rules of standard deontic logic

- Standard deontic logic KD is a normal logic, which means that it provides:
  - K axiom (schema):
  $$\mathbf{O}(\varphi \rightarrow \psi) \rightarrow (\mathbf{O}\varphi \rightarrow \mathbf{O}\psi)$$
  - RN necessitation rule:
  $$\text{If } \vdash \varphi, \text{ then } \vdash \mathbf{O}\varphi.$$

  Rule RN and axiom K define the character of modal possibilities: they obey the rules of logic (and therefore are 'normal'). By RN, logical truths hold in any deontic possibility. By K, the consequences of truths of a deontic possibility are the truths of it.
- The only additional axiom of deontic logic is:
  - D axiom (schema):
  $$\mathbf{O}\varphi \rightarrow \mathbf{P}\varphi$$
- Rule RN can be deontically interpreted as "permission implies logical possibility".[2]
- Deontic interpretation of K is "logical consequences of obligations are obligations themselves".
- Axiom D roughly translates to "it is permitted to fulfil an obligation".

[2]Equating provability without premises with logical necessity RN becomes $\Box\varphi \rightarrow \mathbf{O}\varphi$, and conversion gives suggested reading.

Example

The Roman Law principle *ultra posse nemo obligatur* is also known as *ought implies can* principle, and usually mistakenly attributed to Kant. The principle is translated here in its simplified form (Proposition below): (i) standard deontic logic deals with that which *ought to be* and not, as a full-blown deontic logic should, with that which *ought to be done*, (ii) the alethic modality of logical possibility will be used instead of ability modality.

Lemma

$\vdash \mathbf{P}\varphi \rightarrow \Diamond\varphi$

Proposition

$\vdash \mathbf{O}\varphi \rightarrow \Diamond\varphi$ *(i.e.,* $\vdash \neg\Diamond\varphi \rightarrow \neg\mathbf{O}\varphi$*).*

Proof.

1. Assume $\mathbf{O}\varphi$.
2. $\mathbf{P}\varphi$, from (1) by D.
3. $\Diamond\varphi$, from (2) by lemma.
4. Therefore, $\vdash \mathbf{O}\varphi \rightarrow \Diamond\varphi$.

□

Unexpected results

- The introduction of relational semantics ("possible world semantics", simultaneously and independently discovered in late 1950s by Stig Kanger and Saul Kripke) has brought some amazing insights in philosophy.
- The analogy between quantification, on the one side, and alethic and deontic modality, on the other side, has received its formal semantic explanation.

Many worlds and their relations

- Many worlds. Modal expressions involve hidden quantification: (i) some modalities are universal, like $\Box$ or **O**, and they talk about *all* possibilities (valuations, states, possible worlds) within the appropriate category (logical, deontic,. . .), (ii) some modalities are existential, like $\Diamond$ or **P**, and they in the similar manner talk about *some* possibilities.
- Structure. The plurality of valuations is not sufficient. The distinction between modalities having the same quantificational character but validating different principles lies in the way the possibilities are connected. The possibilities to be taken into account at the point of evaluation. Quantifiers $\forall$ and $\exists$ offer "bird's eye view": their perspective is global and covers all objects. Modalities give a local picture, a"frog's eye view": their perspective is located at a particular evaluation point ("the point where we stand") and therefore covers all possibilities accessible ("visible") from that point.

Example

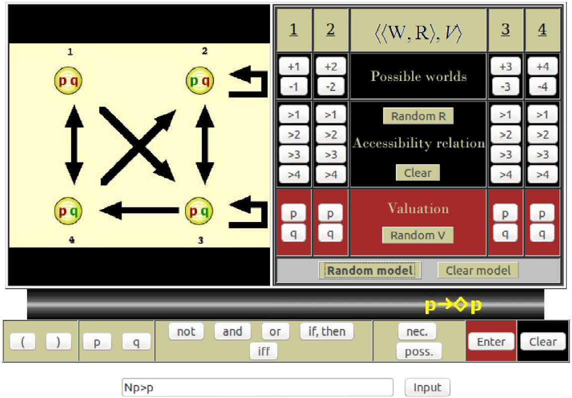
Alethic logic readily accepts the principle of existential modal generalization: 'if something is the case, that it is possible' or $\varphi \to \Diamond\varphi$. Deontic logic readily rejects that principle: it is not valid to claim that 'if something is the case, that it is permitted' or $\varphi \to \mathbf{P}\varphi$.

Geometry of meaning of modal operators

- Intensional semantics takes into account multiple valuations but the meaning of modal operators is not reducible to them. If it were, we could not distinguish the types of operators since the same rule would be associated with deontic **O** operator, alethic $\Box$, epistemic **K**, as well as any other universal modal operator.
- The geometry of meaning of modal operators is given by the properties of accessibility relation. Different modalities have different types of accessibility. The type of accessibility is determined by modal axioms.

  E.g. what property must the relation of deontic accessibility have? The one that corresponds to the meaning of deontic operators, and that meaning is fixed by axioms.
- Thus, the meaning of modal words has two parts:
  1. Rule. Quantification part. Common to all modality types.
  2. Structure. Scope of quantification. Specific for given modality type. Corresponds to axioms of some regional modal logic, and, therefore, exhibits the "geometry of meaning" of the particular modal word.

Modal calculator:
`http://www.ffst.hr/~logika/implog/calculators/modal/modal.html`
Instructions:
`http://www.ffst.hr/~logika/implog/doku.php?id=program:possible_worlds`



Syntax

$$\mathrm{p} ::= \bot \mid p \mid q \mid r \mid \ldots$$
$$\varphi ::= \mathrm{p} \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi.$$

Definition

$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$

Model of unimodal propositional logic

Definition

Model $\mathfrak{M} = \langle\langle W, R\rangle, V\rangle$ comprises:

1. non-empty set (of "possible worlds") W,
2. accessibility relation $R \subseteq W \times W$,
3. valuation V : Propositional letters $\rightarrow \wp W$.

The pair $\langle W, R\rangle$ is called 'frame', and occasionally will be denoted by letter F and indexed.

Truth definition

Definition

(i)	$\mathfrak{M}, w \models p$ iff $w \in V(p)$ for propositional letters p
(ii)	$\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
(iii)	$\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
(iv)	$\mathfrak{M}, w \models \varphi \vee \psi$ iff $\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$
(v)	$\mathfrak{M}, w \models \varphi \rightarrow \psi$ iff $\mathfrak{M}, w \not\models \varphi$ or $\mathfrak{M}, w \models \psi$
(vi)	$\mathfrak{M}, w \models \Box\varphi$ iff $\mathfrak{M}, v \models \varphi$ for all v such that Rwv
(vii)	$\mathfrak{M}, w \models \Diamond\varphi$ iff $\mathfrak{M}, v \models \varphi$ for some v such that Rwv

How to define modal logical truth

Validities

Hierarchy of definitions.

1. (World invariance) φ is *valid*^{1} *in a model* $\mathfrak{M} = \langle W, R, V \rangle$ iff $\mathfrak{M}, w \models \varphi$ for all $w \in W$.
2. (Valuation invariance) φ is *valid*^{2} *on a frame* $F = \langle W, R \rangle$ iff φ is valid^1 for any model $\mathfrak{M}$ over F.
3. (Invariance w.r.t. a set of frames) φ is *valid*^{3} *in a set of frames* S iff φ is valid^2 on every frame $F \in S$.
4. (Full invariance) φ is valid^4 iff φ is valid^2 on every frame F.

A NEW LANGUAGE

The researchers in philosophical logic came upon an amazing insight: for modal axioms there are corresponding properties of accessibility relation. (Axiom K and rule RN are different in category: they define the character of the worlds and say nothing about their connections.) Let's try to introduce this insight by way of a metaphor!

A metaphor

Imagine yourself being repeatedly placed in one world after another within a network of worlds. You have an axiom map: a sentential form that must come out true no matter which sentences you put into it. Your "positive task" is to test the accuracy of the map in the modal way: by looking at accessible worlds, possibly moving there and looking at accessible worlds from there, and possibly repeating the action again but in finitely number of times. It turns out that you come up with positive test results for each of successive placements. After that, you have an additional, more complicated test called "negative task": after being placed in a world you have to investigate whether it is possible to modify the world you are at and the worlds accessible from it so to make the axiom map false. If the positive task always results in affirmative (map is true) and the negative task always gives the negative answer (it is not possible to modify worlds so to falsify the map), then your axiom map is accurate and it describes some property of the paths connecting the worlds. E.g. if the map $\Box\varphi \rightarrow \Box\Box\varphi$ passes both tests in a network of worlds, i.e. if is accurate, then the following fact on the property of paths holds: if you can get from the source world to the target world via an intermediate one, then you can also get directly from the source to the target.

Characterization

Definition

Formula φ characterizes a set of frames S iff φ is valid[2] on every frame $F \in S$ (i.e. $\langle F, V \rangle, w \models \varphi$ for any $F \in S$ and any V and any w).

Characterization means that φ is true everywhere under all valuations in a structure of a given type, and for every structure not being of the given type it is possible to find a valuation and a point where φ is false.

Geometry of meaning and expressive power

- The correspondence between axioms and properties of accessibility relations has revealed an important characteristics of the logic of the language of philosophy and science of man.
- Modal logic is not just another way to define implicitly modal terms by fixing their meaning in axioms. Rather, it is a discovery of a language.
- The language of propositional modal logic turns out to have high expressive power, different in kind from that of the language of propositional logic but lower in discriminatory power from the first-order language.[3]
- The "geometry of meaning" extends far beyond the square or hexagon of oppositions: the logical "space" of modal operators is structured so that different structural types correspond to different modality types.

[3] While the language of propositional logic has no discriminatory power, the language of propositional modal logic can discriminate between finite structures up to bisimilarity. The language of first-order logic can discriminate between finite structures up to isomorphism (a type of "picture relation" stronger than bisimilarity).

Finding correspondences

For some modal formulas (Sahlqvist formulas) the corresponding property of accessibility relation can be computed using Sahlqvist-van Benthem algorithm.[4]

There exists an effective algorithm which translates all modal axioms of the form $A \to B$ into corresponding first-order properties, where A is constructed from basic formulas $\Box \ldots \Box p$ using only $\wedge$, $\vee$, $\Diamond$, B is 'positive': constructed from proposition letters with only $\wedge$, $\vee$, $\Diamond$, $\Box$.

Johan van Benthem (2010)
Modal Logic for Open Minds.
Stanford: CSLI

[4] At `http://www.fmi.uni-sofia.bg/fmi/logic/sqema/new/sqema.jsp` there is an online calculator developed at University of Sofia and implementing the Sahlqvist-van Benthem algorithm.

Standard translation

Definition

$$\begin{aligned}
\mathrm{ST}_x(p) &= Px \\
\mathrm{ST}_x(\bot) &= \bot \\
\mathrm{ST}_x(\neg\varphi) &= \neg\mathrm{ST}_x(\varphi) \\
\mathrm{ST}_x(\varphi \wedge \psi) &= \mathrm{ST}_x(\varphi) \wedge \mathrm{ST}_x(\psi) \\
\mathrm{ST}_x(\Diamond\varphi) &= \exists y(Rxy \wedge \mathrm{ST}_y(\varphi)) \\
\mathrm{ST}_x(\Box\varphi) &= \forall y(Rxy \rightarrow \mathrm{ST}_y(\varphi))
\end{aligned}$$

ST delivers first-order translation. Generalization over all predicates gives the second-order translation $\forall P_1 \ldots \forall P_n \forall x \mathrm{ST}_x(\varphi)$.

Examples

$\mathrm{ST}_x(\mathbf{O}p \rightarrow \mathbf{P}p) = \forall(R_{\mathbf{O}}xy \rightarrow Px) \rightarrow \exists z(R_{\mathbf{O}} \wedge Px)$
$\mathrm{ST}_x(\Box p \rightarrow \Box\Box p) = \forall y(R_N xy \rightarrow Px) \rightarrow \forall y(R_N xy \rightarrow \mathrm{ST}_y(\Box p)) = \forall y(R_N xy \rightarrow Px) \rightarrow \forall y(R_N xy \rightarrow \forall z(R_N yz \rightarrow Pz)$.

The basic idea

The basic idea of Sahlqvist-van Benthem algorithm is to satisfy antecedent in a minimal way.

In the antecedent:				
non-modal or $\Diamond$ occurrences of propositional letters	$Px_1, \dots, Px_n$ by ST	goes to	$(u = x_1 \vee \cdots \vee u = x_n)$	u is the argument of P
sequences of universal modalities of types $i, \dots, j$	$[i] \dots [j]p$	goes to	$R_i \circ \cdots \circ R_j vu$	v is variable obtained by ST_v

Computing correspondence for D axiom

- $\mathbf{O}p \to \mathbf{P}p$, D axiom.
- $\forall P\ \mathrm{ST}_x(\mathbf{O}p \to \mathbf{P}p)$, second-order generalization of the standard translation.
- $\forall P(\forall y(R_{\mathbf{O}}xy \to Py) \to \exists z(R_{\mathbf{O}}xz \wedge Pz))$, inserting standard translation.
- $Pu := R_{\mathbf{O}}xu$, determination of a minimal valuation.
- $\forall y(R_{\mathbf{O}}xy \to R_{\mathbf{O}}xy) \to \exists z(R_{\mathbf{O}}xz \wedge R_{\mathbf{O}}xz)$, replacement of Pu with $R_{\mathbf{O}}xu$.
- $\top \to \exists z(R_{\mathbf{O}}xz \wedge R_{\mathbf{O}}xz)$, simplification.
- $\forall x \exists y R_{\mathbf{O}}xy$, simplification and universal generalization.

Ought implies can—there must be at least one alethic possibility within the scope of indistinguishable obligations

- $\mathbf{O}p \to \Diamond p$
- $\forall P\ \mathrm{ST}_x(\mathbf{O}p \to \Diamond p)$
- $\forall P(\forall y(R_{\mathbf{O}}xy \to Py) \to \exists y(R_{\mathrm{N}}xy \wedge Py))$
- minimal valuation $Pu := R_{\mathbf{O}}xu$
- $\forall y(R_{\mathbf{O}}xy \to R_{\mathbf{O}}xy) \to \exists y(R_{\mathrm{N}}xy \wedge R_{\mathbf{O}}xy)$
- $\top \to \exists y(R_{\mathrm{N}}xy \wedge R_{\mathbf{O}}xy)$
- $\forall x \exists y(R_{\mathrm{N}}xy \wedge R_{\mathbf{O}}xy)$

"convergent mixed seriality".

Ought implies can Chellas reading—deontic possibilities operate within the scope of alethic possibilities

- (Ch) $\Box p \to \mathbf{O}p$
- $\forall P\ \mathrm{ST}_x(\Box p \to \mathbf{O}p)$
- $\forall P(\forall y(R_{\mathrm{N}}xy \to Py) \to \forall y(R_{\mathbf{O}}xy \to Py))$
- minimal valuation $Pu := R_{\mathrm{N}}xu$
- $\forall y(R_{\mathrm{N}}xy \to R_{\mathrm{N}}xy) \to \forall y(R_{\mathbf{O}}xy \to R_{\mathrm{N}}xy)$
- $\top \to \forall y(R_{\mathbf{O}}xy \to R_{\mathrm{N}}xy)$
- $\forall x \forall y(R_{\mathbf{O}}xy \to R_{\mathrm{N}}xy)$

Expressive power of the language of propositional modal logic

The language of propositional modal logic is more expressive than the language of propositional logic: the latter cannot discriminate structures, the former can. The question arises as to how close can it get to its descriptum?

Definition

Bisimulation is a relation E between structures $\mathfrak{M} = \langle W, R, V \rangle$ and $\mathfrak{M}' = \langle W', R', V' \rangle$ such that

$$wEw'$$

iff:

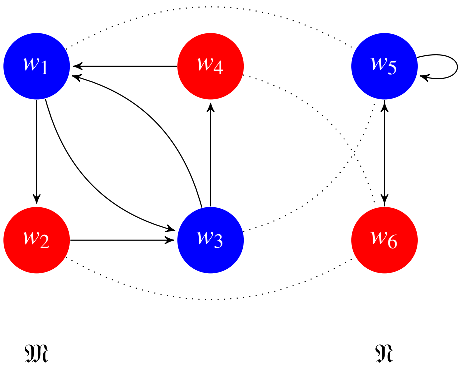
atomic harmony $w \in V(p) \leftrightarrow w' \in V'(p)$ for any propositional letter p,

"forth" $Rwv \rightarrow \exists v'(vEv' \wedge R'w'v')$,

"back" $R'w'v' \rightarrow \exists v(vEv' \wedge Rwv)$.

Structural similarity

Bisimulation is a more general (or weaker) concept of similarity than isomorphism: (i) bijection is not required, (ii) the points are indistinguishable if they carry identical atomic information along similar paths. In a metaphor: —the language of first-order logic gives a perspective from above, a "bird's eye view"; —The language of propositional modal logic gives a perspective from below, a "frog's eye view".



Relations R and R' are represented by arrows. Dotted lines stand for bisimulation relation E. The points w and w' have identical colors if their valuations coincide (i.e., $w \in V(p)$ iff $w' \in V'(p)$ for any letter p). Since bisimulation is not bijective it is possible that $w \neq v$, Rwv, Rvw, wEw', vEv' while $w' = v'$ and $R'w'w'$. The picture shows how "frog's eye view" cannot distinguish between "symmetrical" and "reflexive" points.

Bisimilar models are modally indistinguishable

Proposition (Bisimilarity implies modal harmony)

If a relation E is a bisimulation between $\mathfrak{M} = \langle W, R, V\rangle$ and $\mathfrak{M}' = \langle W', R', V'\rangle$ and wEw', then for any sentence φ of basic propositional modal logic it holds that

$$\mathfrak{M}, w \models \varphi \text{ iff } \mathfrak{M}', w' \models \varphi.$$

Proof.

In the basic case $\mathfrak{M}, w \models p$ iff $\mathfrak{M}', w' \models p$ because of "atomic harmony" between w and w' (i.e., they verify the same propsitional letters).
For inductive case let us consider only $\Diamond\varphi$ as an example. By inductive hypothesis for proper subformulas of the formula under investigation the fact to be established holds (here—modal harmony holds for φ).

1. $\mathfrak{M}, w \models \Diamond\varphi$ (assumption for the L-R direction)
2. There is a v s.t. Rwv and $\mathfrak{M}, v \models \varphi$. (by truth definition)
3. There is a v' s.t. vEv' and $R'w'v'$. (by "forth" condition)
4. $\mathfrak{M}', v' \models \varphi$ (by inductive hypothesis)
5. $\mathfrak{M}', w' \models \Diamond\varphi$ (by truth definition, from 3 and 4)

The R-L direction is similar but in this case we rely on "back condition" of E. □

Bisimilarity and modal theories

- Bismilar models verify the same formulas. Does the converse hold? Yes if the models are finite.

If $\mathfrak{M}$ and $\mathfrak{M}'$ are finite models such that $\mathfrak{M}, w \models \varphi \leftrightarrow \mathfrak{M}', w' \models \varphi$ for all $\varphi \in \mathcal{L}_{ML}$, then there is a bisimulation E such that wEw'.

If $\mathfrak{M}$ and $\mathfrak{M}'$ are finite models such that $\mathfrak{M}, w \models \varphi \leftrightarrow \mathfrak{M}', w' \models \varphi$ for all $\varphi \in \mathcal{L}_{ML}$, then there is a bisimulation E such that wEw'.

Proof.

At. harmony Atomic harmony is a special case of the proposition condition.

Forth Assume: wEw' and Rwv. To prove: $\exists v'(vEv' \wedge R'w'v')$.

1. $\forall v'(R'w'v' \to \neg vEv')$ (reductio assumption).
2. Define $S' = \{u' \mid R'w'u'\}$ (separation of a subset of W').
3. $\mathfrak{M}, w \models \Diamond\top$ (since Rwv).
4. $\mathfrak{M}', w' \models \Diamond\top$ (because of modal equivalence of bisimilarity).
5. $S' \neq \emptyset$ (from 4, semantics).
6. $S' = \{u'_1, \ldots, u'_n\}$ (since models are finite).
7. For each $u'_i \in S'$ there is a distinguishing sentence φ_i true at v (i.e. $\mathfrak{M}, v \models \varphi_i$) but false at u'_i (i.e. $\mathfrak{M}', u'_i \not\models \varphi_i$) (from 1 and modal equivalence of bisimilarity).
8. $\mathfrak{M}, w \models \Diamond(\varphi_i \wedge \cdots \wedge \varphi_n)$ (semantics, Rwv and 7).
9. $\mathfrak{M}', w' \not\models \Diamond(\varphi_i \wedge \cdots \wedge \varphi_n)$ (semantics, 6,7).
10. $\mathfrak{M}', w' \models \Diamond(\varphi_i \wedge \cdots \wedge \varphi_n)$ (since wEw')
11. $\perp$

Back Similar to "forth".

□

From the literature

Viewed as tools for defining frames, every modal formula corresponds to a second-order formula. Although this second-order formula sometimes has a first-order equivalent, even quite simple modal formulas can define classes of frames that no first-order formula can. In spite of this, there are extremely simple first-order definable frame classes which no modal formula can define. In short, viewed as frame description languages, modal languages exhibit an unusual blend of first- and second-order expressive powers.

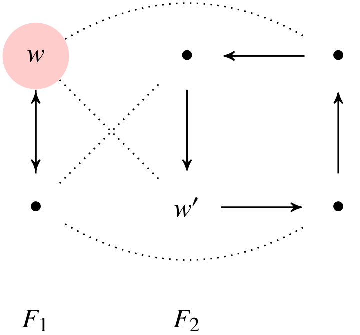
P. Blackbourn, M. de Rijke, Y. Venema.
Modal Logic
Cambridge University Press, 2002.

Proposition

No modal formula characterizes asymmetric frames.

Proof.

Suppose that φ is a modal formula that characterizes asymmetric frames. Since F_1 is symmetric it is possible to construct a model $\mathfrak{M} = \langle F_1, V \rangle$ for some valuation V such that point w provides a counterexample for φ, i.e., $\mathfrak{M}, w \not\models \varphi$. A bisimulation E can be established between $\mathfrak{M} = \langle F_1, V \rangle$ and $\mathfrak{N} = \langle F_2, V' \rangle$ for some V'. Since φ characterizes asymmetric frames it must be true at w', i.e., $\mathfrak{N}, w' \models \varphi$. Since wEw', by modal equivalence of bisimilar model it must hold that $\mathfrak{N}, w' \not\models \varphi$. $\perp$



□

Labeled natural deduction

Basin, Matthews and Vigano have developed a system of deduction for normal logics with one modality lying within "Geach's hierarchy" (i.e. those whose relational theory is representable in first order language). [5] The rules are the same for any universal and existential modality; the differences between logics are introduced using relational theory which describes frame properties. At each w we use classical rules except for negation , whose introduction rule is "global": $\Gamma, w : \varphi \vdash v : \bot \Rightarrow \Gamma \vdash w : \neg\varphi$.

- Rules for operators)

$\Box$Intro	$\Gamma, Rwv \vdash v : \varphi \Rightarrow \Gamma \vdash w : \Box\varphi$	v does not occur in Γ
$\Box$Elim	$\Gamma \vdash Rwv,\ w : \Box\varphi \Rightarrow \Gamma \vdash v : \varphi$	
$\Diamond$Intro	$\Gamma \vdash Rwv,\ v : \varphi \Rightarrow \Gamma \vdash w : \Diamond\varphi$	
$\Diamond$Elim	$\Gamma, Rwv,\ v : \varphi \vdash \psi \Rightarrow \Gamma, w : \Diamond\varphi \vdash \psi$	v does not occur in $\Gamma \cup \{\psi\}$

- Relational theory follows from frame correspondences for specific modalities. As an example here a relation theory for deontic and alethic modalities is given.

(D) $\vdash R_{\mathbf{O}}(w, f(w))$
(T) $\vdash R_{\mathrm{N}}(w, w)$
(4) $R_{\mathrm{N}}(w,v), R_{\mathrm{N}}(v,u) \vdash R_{\mathrm{N}}(w,u)$
(B) $R_{\mathrm{N}}(w,v) \vdash R_{\mathrm{N}}(v,w)$
(Ch) $R_{\mathbf{O}}(w,v) \vdash R_{\mathrm{N}}(v,w)$

[5] Extended to polymodal logic by Žarnić (2006).

DERIVING *ought* FROM *is*

Lemma (Stoic)

If something is necessary, then it ought to be.[a]

[a] Compare Spinoza's Proposition 37 (V) in *Ethics*: "There is nothing in nature, which is contrary to this intellectual love, or which can take it away."

Proof.

1	$w : \Box p$	
2	$v \quad R_{\mathbf{O}}wv$	
3	$R_{\mathrm{N}}wv$	2/ Ch
4	$v : p$	1, 3/ Elim$\Box$
5	$w : \mathbf{O}p$	2–4/ Intro$\mathbf{O}$
6	$w : \Box p \rightarrow \mathbf{O}p$	1–5/ Intro$\rightarrow$

□

Proposition

Determinism causes deontic collapse.

Proof.

1	$w : p \to \Box p$	
2	$w : \neg p \to \Box\neg p$	
3	$w : p$	
4	$w : \Box p$	1, 3/ Elim$\to$
5	$w : \mathbf{O}p$	4/ Stoic
6	$w : \mathbf{O}p$	
7	$R_{\mathbf{O}}wfw$	D
8	$R_{\mathrm{N}}wfw$	7/ Ch
9	$w : \neg p$	
10	$w : \Box\neg p$	2, 9/ Elim$\to$
11	$fw : p$	6, 7/ Elim$\mathbf{O}$
12	$fw : \neg p$	8, 10/ Intro$\Box$
13	$w : p$	9–12/ Elim$\neg$
14	$w : p \leftrightarrow \mathbf{O}p$	3–5, 6–13/ Intro$\leftrightarrow$

□

Tarskian consequence

Axiom 1. $|S| \leq \aleph_0$.
Axiom 2. If $X \subseteq S$, then $X \subseteq Cn(X) \subseteq S$.
Axiom 3. If $X \subseteq S$, then $Cn(Cn(X)) = Cn(X)$.
Axiom 4. If $X \subseteq S$, then $Cn(X) = \bigcup_{Y \subseteq X \text{ and } |Y| < \aleph_0} Cn(Y)$.
Axiom 5. There exists a sentence $x \in S$ such that $Cn(\{x\}) = S$.

Alfred Tarski. 1928. On some fundamental concepts of metamathematics.
In Alfred Tarski. *Logic, semantics, metamathematics : papers from 1923 to 1938* (trans. by J.H.Woodger), Clarendon Press, Oxford, 1956

Tarski's general axioms of consequence relation, construed as the relation between sets of sentences $Cn \subseteq \wp S \times \wp S$, could be expressed in the natural language as follows:

1. For countable languages S (Axiom 1) it holds that:
2. consequences of sentences remain within the same language and premises are their own consequences (reflexivity; Axiom 2),
3. consequences of consequences of a set are already consequences of that set (transitivity; Axiom 3),
4. consequences of a set X do not exceed the consequences of their finite subsets Y, which are retained in their superset X consequences (compactness and monotonicity, Axiom 4),
5. there is at least one sentence in the language such that its consequences include all the sentences of that language (existence of *falsum*, "absurdity," "explosive sentence," "informational breakdown," etc.; Axiom 5).

Geach's description

Some years ago I read a letter in a political weekly to some such effect as this. 'I do not dispute [X's] premises, nor the logic of his inference. But even if a conclusion is validly drawn from acceptable premises, we are not obliged to accept it if those premises are incomplete; and unfortunately there is a vital premise missing from [X's] argument—the existence of [Y].' I do not know what [X's] original argument had been; whether this criticism of it could be apt depends on whether it was a piece of indicative or of practical reasoning. Indicative reasoning from a set of premises, if valid, could of course not be invalidated because there is a premise "missing" from the set. But a piece of practical reasoning from a set of premises can be invalidated thus: your opponent produces a *fiat* you have to accept, and the addition of this to the *fiats* you have already accepted yields a combination with which your conclusion is inconsistent.

Peter Geach.
Dr. Kenny on practical inference.
Analysis (1966) **26**: 76–79

Defeasibility of conclusion and completeness of premises

The consequence relation described by Geach has two notable properties:

- ("locality") conclusion holds in virtue of premises but it can be defeated by additional premises;
- (existence of the limit) if the premises are complete the conclusion cannot be defeated (where 'conclusion is defeated' means 'premises are acceptable and conclusion is not acceptable').
- By 'Geach's problem' I mean a problem of devising modeltheoretic notion of consequence relation that captures the pretheoretical notions of conclusion defeasibility and of "completeness of premises".

Information containment conception of logical consequence

Information

Whenever i **L**-implies j, i asserts all that is asserted by j, and possibly more. In other words, the information carried by i includes the information carried by j as a (perhaps improper) part. Using '$In(\ldots)$' as an abbreviation for the presystematic concept 'the information carried by ...,' we can now state the requirement in the following way:
R3-1. $In(i)$ includes $In(j)$ iff i **L**-implies j.
By this requirement we have committed ourselves to treat information as a set or class of something. This stands in good agreement with common ways of expression, as for example, "The information supplied by this statement is more inclusive than (or is identical with, or overlaps) that supplied by the other statement."

Rudolf Carnap and Yehoshua Bar-Hillel. *An Outline of a Theory of Semantic Information*. Technical Report no. 247. Research Laboratory of Electronics, Massachusetts Institute of Technology, 1952.

Adding information

- Two notions "adding information" and "information as a set or class of something" show
  1. that sentences can do something, namely they can "add information", and
  2. that semantic relations occur at the level of sets, since "information [is] a set or class of something".
- Putting these two together we get that sentences act on sets.
- Two notions of "information containment" are relevant:
  1. Conclusion adds no information to **any** context that includes all information contained in premises.
  2. Conclusion adds no information to the context that includes **only** the information contained in premises.[This notion corresponds to "ignorant-update-to test" and a variant of called *prima facie* consequence is suitable for practical logic (here practical logic is used in Aristotelian sense as opposed to theoretical logic).]

The simplest update semantics for propositional logic

- The basic ideas: information is a set of valuations, valuations are sets of propositional letters, sentences are functions taking a set of valuations and delivering a set of valuations.
- The set of all valuations $W = \wp A$ with respect to the set A of propositional letters.
- An informational state is identified with set of valuations $\sigma \subseteq W$. Here, "less is more": lesser the number of valuations w in σ, greater the amount of information in σ.
- Limit cases:

  Minimal info-state if $\sigma = W$ (i.e. $|\sigma| = |W|$), then σ contains no information;
  Maximal info-state if $|\sigma| = 1$, σ gives full information w.r.t. A;
  Absurd info-state if $\sigma = \emptyset$ (i.e. $|\sigma| = 0$), σ shows that learning (information acquisition process) has failed.

Update functions

- Truth in a valuation: for $w \in W$, $p \in A$, $\varphi, \psi \in \mathcal{L}_A$
  $w \models p$ iff $p \in w$,
  $w \models \neg\varphi$ iff $w \not\models \varphi$,
  $w \models (\varphi \wedge \psi)$ iff $w \models \varphi$ and $w \models \psi$.
- Update function [6]: $\cdot[\cdot] : \wp W \times \mathcal{L}_A \rightarrow \wp W$.
- Update-sentences φ^+ act upon info-states delivering info-state in which they are accepted:
- $\sigma[\varphi^+] = \{w \in \sigma \mid w \models \phi\}$,
- $\sigma[\varphi^+] = \sigma[\varphi^+][\varphi^+]$.
- Use calculator!

[6] Combined infix and postfix notation is used.

Adding and testing

- Relative consistency testing (can an information contained in φ be added in a context σ without causing informational breakdown)
    - Acceptability testing:

$$\sigma\left[\varphi^{?consistency}\right] = \begin{cases} \sigma & \text{if } \sigma\left[\varphi\right] \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

- Relative validity testing (will the context σ be changed by adding information contained in φ)
    - Acceptance testing:

$$\sigma\left[\varphi^{?validity}\right] = \begin{cases} \sigma & \text{if } \sigma\left[\varphi\right] = \sigma, \\ \emptyset & \text{otherwise.} \end{cases}$$

Classical consequence as a special case

In dynamic semantics the notion of consequence can be generalized. The use of 'therefore φ' is justified in context σ if φ produces no change in σ: $\sigma[\varphi] = \sigma$. Then special cases of relations between text and sentence can be distinguished: is the text order irrelevant (tt) or not (ut), is the relation "localized" (0-ut), etc.

---

test-to-test $p_0; ...; p_n \models_{tt} q$ iff for all contexts σ:

$$\sigma[p_1] = ... = \sigma[p_n] = \sigma \to \sigma[q] = \sigma$$

update-to-test $p_0; ...; p_n \models_{ut} q$ iff for all contexts σ:

$$\sigma[p_1]...[p_n] = \sigma[p_1]...[p_n][q]$$

ignorant-update-to-test $p_0; ...; p_n \models_{0-ut} q$ iff for the empty context (carrying no information) 0:

$$0[p_1]...[p_n] = 0[p_1]...[p_n][q]$$

Johan van Benthem. 1996. *Exploring Logical Dynamics*
Stanford: Center for the Study of Language and Information

Enriched language

Example

In the language enriched with “modalities of introspection” the structural properties of classical (i.e. Tarskian) consequence relation do not hold.

- **irreflexivity** might $\neg p$; $p \not\models_{ut}$ might $\neg p$,
- **non-monotonicity** might $p \models_{ut}$ might p, but might p, $\neg p \not\models_{ut}$ might p,
- **non-transitivity** $p, \neg q \models_{0-ut} p \vee q$ and $p \vee q \models_{0-ut}$ might q, but $p, \neg q \not\models_{0-ut}$ might q

A paradoxical imperative inference

1 Slip the letter into the letter-box!
2 Slip the letter into the letter-box or burn it!
3 You may: slip the letter into the letter-box or burn it.
4 You may: burn the letter.
5 Therefore, if you ought to slip the letter into the letter box, then you may burn it.

	(Purportedly) holds in virtue of	*Intuitive acceptability*
1		
2	meaning of 'or'; from 1	ambivalent
3	relations between 'must' and 'may'; from 2	affirmative
4	distributivity of "free choice permission"; from 3	mainly affirmative
5		negative

- Unexpected behavior of 'or' in 2 and 4.

PARADOXICAL INFERENCE AGAIN: A DEONTIC VARIANT

0	$p \models p \vee q$	meaning of $\vee$
1	$\mathrm{O}p \models \mathrm{O}(p \vee q)$	Scott's principle
2	$\mathrm{O}(p \vee q) \models \mathrm{P}(p \vee q)$	D axiom
3	$\mathrm{O}p \models \mathrm{P}(p \vee q)$	by $\models$ transitivity; from 1, 2
4	$\mathrm{P}(p \vee q) \models \mathrm{P}q$	by free choice permission
5	$\mathrm{O}p \models \mathrm{P}q$	by $\models$ transitivity; from 3, 4

- The consequence relation 1, which is intuitively less plausible than 4, holds in normal deontic logic while 4 does not hold.
- Scott's principle

$$\{(p_1 \wedge ... \wedge p_{n-1}) \rightarrow q\} \vdash (\Box p_1 \wedge ... \wedge \Box p_{n-1}) \rightarrow \Box q$$

($n \geq 1$) characterizes normal propositional modal logic (e.g. it may replace K axiom and neccessitation rule). It may be read as stating that "meaning relations" of propositional logic, i.e. meaning relations holding in virtue of meaning of truth-functional connectives, are preserved in the modal context.

Avoiding the paradox

- (Alf) Ross' paradox and free choice permission show that logical terms may "change their behavior" in the presence of other logical terms.
    - The odd result that if anything is obligatory than everything is permitted (i.e. Op ⇒ Pq) shows that one may have intuitions that confirm isolated consequence steps and still lack the intuition that confirms transitive closure of these steps.
- The pretheoretical understanding of logical relations may well be holistic in character: perhaps there is no unique understanding of logical terms that is *constitutive* for the understanding of consequence relations, and perhaps there is no unique understanding of admissible consequence relations that is *regulative* for the understanding of logical terms.
- It seems very clear for me that we should not repeat Kant's mistake and accept classical logic as the final word in philosophy and science. Modal logic and its extensions have made it possible for us to enter the realm of complex informational, mental, normative, and social phenomena. It also shows that logic goes beyond disciplinary fragmentation of knowledge and communicative closure of academic communities: it belongs to everyone, not as a "plug and play" device, but as common type of research task.